Limitations and failures of the Layzer model for hydrodynamic instabilities

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We report several limitations and failure modes of the recently expanded Layzer model for hydrodynamic instabilities. The failures occur for large initial amplitudes, for stable accelerations, and for spikes in two-fluid systems.

Hydrodynamic instabilities at fluid interfaces continue to be extensively studied theoretically, computationally, and experimentally [1] because they induce significant mixing. Best known is the deleterious effect of mixing on thermonuclear burn in inertial confinement fusion (ICF) capsules [2]. Supernova explosions also display mixing of inner shells [3]. Any interface that is shocked or accelerated is subject to Richtmyer-Meshkov (RM) or Rayleigh-Taylor (RT) instabilities. Perturbations of amplitude η and wave number k at the interface between two fluids of densities ρ_A and ρ_B undergoing acceleration g evolve according to

$$\ddot{\eta} - gkA\,\eta = 0. \tag{1}$$

Here the Atwood number A is defined as $(\rho_B - \rho_A)/(\rho_B + \rho_A)$. This equation is valid in the linear regime, $\eta k \ll 1$. In the nonlinear or large-amplitude regime the most commonly used model is the Layzer model [4]. It has recently been extended [5–8] and applied to experiments [9,10] and compared with simulations [11]. In this paper we point out the limitations and failures of this model. We hope to spur further extension of the model and, more importantly, prompt scrutiny of other models [12–14] where the issues we raise here have not been addressed at all.

We find the following: (1) The model fails for initial amplitudes $\eta_0 \gtrsim (\eta_0)_{\max}$—see Eq. (8). (2) The model fails for a large class of acceleration histories $g(t)$—see Eq. (10). (3) The model fails to describe spikes for $A < 1$. For $A = 1$, however, the model for spikes is even more robust than for bubbles for which it was originally proposed—see Fig. 4.

Notation. We take the interface to be given by $\eta(t) + \eta_2(t)x^2$, where x is the coordinate along the interface. Higher-order terms are neglected in the model. The initial shape of the interface is given by $\eta_0 \cos(kx)$ for two dimensions (2D) and $\eta_0 J_0(\beta_1 r/R)$ for 3D, where J_0 (J_1) is the Bessel function of order zero (one) and β_1 is the first zero of J_1, $\beta_1 \approx 3.832$. We introduce the parameter c with $c = 2$ for 2D and $c = 1$ for 3D, and take $k = 2\pi/\lambda$ for 2D and $k = \beta_1/R$ for 3D. Expanding the cosine and Bessel functions we find

$$\eta_2(0) = -ck^2\eta_0/4, \tag{2}$$

where $\eta_0 \equiv \eta(t=0)$ is the initial amplitude. With this notation the linear result is independent of c and valid for both 2D and 3D.

The model. Layzer proposed his nonlinear RT model [4] for $A = 1$, $\eta_0 = 0$, and constant acceleration. The model being incompressible, the RM instability is treated as $g = 0$ with the

shock providing $\dot{\eta}_0$ to initialize the problem. In [5] we extended the model to arbitrary η_0. The equations can be written as

$$(2\eta_2 + ck/2)\ddot{\eta} + c^2k^2\dot{\eta}^2/4 + 2g\,\eta_2 = 0, \tag{3}$$

$$\eta_2(t) = -ck\{1 + [(1+c)\eta_0 k - 1]e^{-k(1+c)(\eta-\eta_0)}\}/4(1+c), \tag{4}$$

where Eq. (2) has been used for $\eta_2(0)$. As a check, define $\theta(t) = e^{(\eta-\eta_0)k}$ to obtain Eqs. (2a) and (2b) in [5] for $c = 2$ (2D) and $c = 1$ (3D), respectively. For arbitrary η_0 we obtained a first integral for $g = $ const and a second integral, i.e., full analytic solution, for $g = 0$. We pointed out particularly simple solutions for $\eta_0 = \eta^*$ defined by

$$\eta^* \equiv 1/k(1+c). \tag{5}$$

Zhang [6] considered 2D geometry only and his equations and solutions for bubbles agreed with ours. He also proposed a model for spikes and found fair agreement with previous calculations.

An extension to arbitrary A was achieved by Goncharov [7]. His results can be written as

$$F_1\frac{\ddot{\eta}}{D} + F_2\frac{c^2k^2\dot{\eta}^2}{8D^2} + 2gA\,\eta_2 = 0 \tag{6}$$

with

$$F_1 = 2A\,\eta_2^2 + c^2Ak\,\eta_2/2(1+c) - c^2k^2/8(1+c), \tag{7a}$$

$$F_2 = 2A\,\eta_2^2 + (A + cA - 2c - 1)k\,\eta_2/(1+c) \\ + ck^2(3cA/2 + A - c - 1)/4(1+c)^2, \tag{7b}$$

$$D = \eta_2 - ck/4(1+c), \tag{7c}$$

and η_2 still given by Eq. (4). In addition, he proposed a model for spikes, but found that at moderate A the model underestimates η^{spike}.

We pointed out [8] that a transformation generalizes our earlier $A = 1$ results to arbitrary A provided $\eta_0 = \eta^*$ and proposed a simple analytic model for the evolution of RT and RM bubbles from the linear to the nonlinear regime.

Bubbles. Depending on the numerical scheme used to solve the second-order ordinary differential equation (ODE) [Eq. (6)] we find that either the solution fails or gives a patently wrong answer for $\eta_0 > (\eta_0)_{\max}$ given by
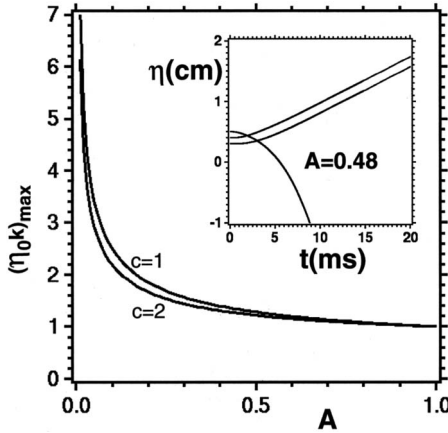
FIG. 1. $(\eta_0 k)_{max}$ vs A from Eq. (8). The Layzer model fails for $\eta_0 \geq (\eta_0)_{max}$. The inset shows RT bubble amplitudes with $\eta_0 = 0.3$, 0.4, and 0.5 cm, the last one plainly wrong.

$$(\eta_0 k)_{max} = \frac{c}{2(1+c)}\left(1 + \sqrt{1 + \frac{4(1+c)}{Ac^2}}\right) \qquad (8)$$

and plotted in Fig. 1. The inset in that figure displays $\eta(t)$ for $\eta_0 = 0.3$, 0.4, and 0.5 cm in a *gedanken* linear electric motor (LEM) experiment [15] discussed in Ref. [8]: $g = 70 g_{Earth}$, $\lambda = 7.3/3$ cm, 7.3 cm being the width of the tank containing hexane as the light fluid and a water-NaI solution for the heavy fluid ($A \sim 0.48$). One does not need numerical simulations (although we did perform several) to declare the $\eta_0 = 0.5$ cm solution patently wrong. Indeed, for $c = 2$, $A = 0.48$, and $k = 6\pi/7.3 \sim 2.58$ cm^{-1}, Eq. (8) gives $(\eta_0)_{max} \approx 0.48$ cm, explaining the failure at $\eta_0 = 0.5$ cm.

Equation (8) is obtained by analyzing F_1, the coefficient of $\ddot{\eta}$ in Eq. (6), after writing it as $F_1 = 2A(\eta_2 - \eta_2^+)(\eta_2 - \eta_2^-)$ with $\eta_2^\pm = c^2 k\{-1 \pm [1 + 4(1+c)/Ac^2]^{1/2}\}/8(1+c)$ and finding for what values of $\eta_2(0)$ can this coefficient vanish, using the fact that $\eta_2(t)$ varies between $\eta_2(0)$ given by Eq. (2) and $\eta_2(\infty) = -ck/4(1+c)$ from Eq. (4). For example, for $A = 1$ the coefficient of $\ddot{\eta}$ in Eq. (3) at $t = 0$ is $ck(1 - \eta_0 k)/2$ and, at $t = \infty$, it is $c^2 k/2(1+c)$, which is positive. Clearly, at some time the coefficient will vanish unless $\eta_0 k < 1$, which is indeed $(\eta_0 k)_{max}$ for $A = 1$ in Eq. (8), independent of c.

While $\eta_0 \geq (\eta_0)_{max}$ guarantees failure, $\eta_0 < (\eta_0)_{max}$ does not guarantee success. There is a large body of evidence [4–11] confirming that the model does well for RT and RM instabilities, and we quote the asymptotic bubble velocities:

$$\dot{\eta}(\infty) = 2\sqrt{\frac{gA}{c(1+c)(1+A)k}}, \quad \text{RT},$$

$$\dot{\eta}(\infty) = \frac{2(1+c+cA-A)}{c(1+c)(1+A)kt}, \quad \text{RM}. \qquad (9)$$

Hence we ask the question: Assuming that we start with an amplitude in the admissible region, i.e., $\eta_0 < (\eta_0)_{max}$, does the Layzer model give the correct answer for arbitrary acceleration histories?

It is well known that for $\eta k \ll 1$ the Layzer model reduces to the linear result, Eq. (1), and that this equation is valid for

arbitrary $g(t)$. We have just shown that the model fails for $\eta_0 \geq (\eta_0)_{max}$. Since its lower limit is trivial and its upper limit is prohibited, we therefore propose attacking the problem with an intermediate value. Between $\eta_0 \sim 0$ and $\eta_0 \sim (\eta_0)_{max}$ lies the value $\eta_0 = \eta^* = 1/k(1+c)$. From Fig. 1 and the above discussion it is clear that η^* lies in the admissible region for *all A*.

Now, for $\eta_0 = \eta^*$ Eq. (4) gives a constant η_2 (this is the reason for the exceptionally simple solutions given in [5,8]) and Eq. (6) can be written as

$$\ddot{\theta}_L - gk_L A_L \theta_L = 0, \qquad (10)$$

where $k_L = c(1+c)(1+A)k/2(1+c+cA-A)$, $A_L = 2A/(1+c+cA-A)$, and $\theta_L = e^{(\eta - \eta_0)k_L}$. Note the similarity of Eq. (10) with Eq. (1): Any solution to the linear equation automatically gives a solution to Eq. (10) with the replacement of k with k_L, A with A_L, and η with θ_L. The nonlinear solution is essentially the logarithm of the linear solution.

An important limitation can be deduced immediately: Any $g(t)$ that yields a sign-changing $\eta_{linear}(t)$ must be excluded because θ_L, the nonlinear solution for the same $g(t)$, is positive definite and cannot change sign. We found this by accelerating the LEM tank upward, i.e., in the stable direction, and found that the Layzer model fails to produce the oscillating gravity waves seen in the simulations and, of course, expected on physical grounds as the phenomenon was well known long before the RT instability. Though it helps to obtain Eq. (10) and understand the mathematical origin by setting $\eta_0 = \eta^*$, this is not necessary; solutions to Eq. (3) or (6) with any other η_0, which must be obtained numerically, also reveal this failure.

A possible solution may be to switch back and forth between bubble and spike equations (see below) every time η goes through zero. We do not pursue such an approach here.

Another useful, but probably weaker argument follows from Eq. (4): Since $\eta_2(t) = \text{const}$ for $\eta_0 = \eta^*$, how can $\eta(t)$ change sign or oscillate while keeping the same initial curvature? We conclude that the Layzer model is primarily for acceleration histories that allow the maintenance of the initial curvature—no phase changes. Indeed $\eta(t)$ for RT and RM instabilities grows with uniform sign and, as long as $\eta_0 < (\eta_0)_{max}$, the Layzer model successfully predicts their evolution (note that an application to a phase-reversing experiment was done *after* the reversal [10]). Since η_{RM} and η_{RT} grow logarithmically and linearly, respectively, with time, we looked for a quadratically growing $\eta(t)$. An analytic solution to Eq. (10) is easily obtained with the proper choice of constants. Let $g(t) = g(0)(1 + \alpha t^2)$ and take $\eta(t) = \eta_0(1 + \beta t^2)$ with $\eta_0 = \eta^*$. One finds $\beta = \alpha/2k_L \eta_0 = g(0)A_L/2\eta_0$. We applied this acceleration to the LEM tank taking $g(0) = 10 g_{Earth} = 0.0098$ cm/ms^2 and $\lambda = 2.43$ cm, $A = 0.48$. The result is shown in Fig. 2 for $\eta_0 = \eta^*$, $2\eta^*$, and $4\eta^*$, i.e., $\eta_0 = 0.13$, 0.26, and 0.52 cm. We used the above analytic equation for the first ($\eta_0 = \eta^*$) run, and Eq. (6) for the others. The last solution fails because $4\eta^* > (\eta_0)_{max}$. The dashed lines in Fig. 2 are CALE [16] simulations showing essentially quadratic bubble growth for all η_0. Needless to say, there are no physical reasons why the initial amplitude cannot be large
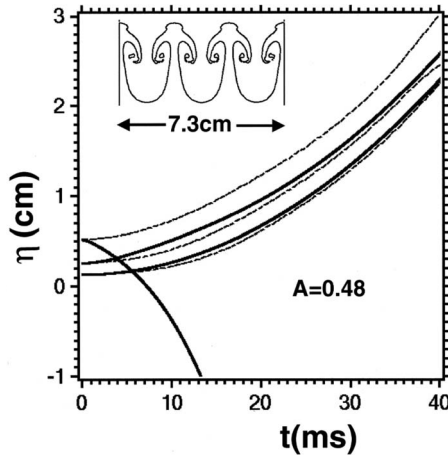
FIG. 2. Bubble amplitudes vs time for a quadratically increasing $g(t)$—see text. Continuous lines are from the model and the dashed lines from CALE simulations. The inset shows the interface at 30 ms for the $\eta_0 = 0.13$ cm run.
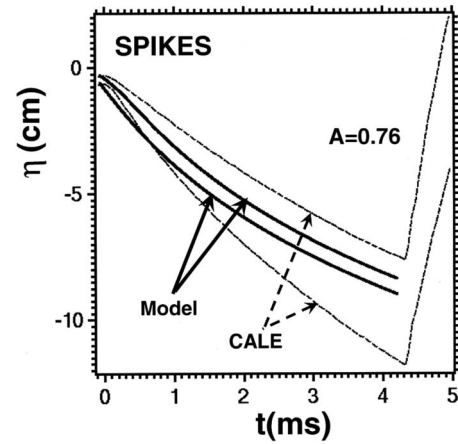

FIG. 3. RM spike amplitudes vs time from the Layzer model and CALE simulations. $M_s = 1.2$ shock strikes a perturbed He-air interface. The model overestimates the small-amplitude run ($\eta_0^{spike} = -0.35$ cm) and underestimates the large-amplitude run ($\eta_0^{spike} = -0.70$ cm). Reshock occurs at 4.2 ms.

and the hydrocode appears to produce reasonable results (there are no experiments of this type)—only the model fails.

Spikes. Despite Layzer's warnings about applying the model to the "flow near the walls," both Zhang and Goncharov proposed models for spikes. Limited to $A=1$ and $c=2$, Zhang proposed [6] using the *same* equations except with an initially negative η_0 [and a positive $\eta_2(0)$—see Eq. (2)]. On the other hand, Goncharov proposed [7] the transformation $\eta \rightarrow -\eta$, $\eta_2 \rightarrow -\eta_2$, $A \rightarrow -A$, and $g \rightarrow -g$. Both approaches correctly reproduce the linear result, Eq. (1). However, *they differ in the nonlinear regime*. We find Goncharov's model for spikes to be in error in practically all cases that we studied. The asymptotic spike velocities can be obtained by applying the above transformation to Eq. (9); thus one obtains equations also found in drag-buoyancy models [17,18]. Despite claims that such equations [Eq. (9) with $g \rightarrow -g$, $A \rightarrow -A$] agree with simulations, we believe these are accidental. Goncharov already pointed out the shortcoming of his model for the RT spike. Our own simulations with $c=1$ or 2 show the same trend: The RT spike is underestimated by such a model. As for RM, we find that the model overestimates simulations with small η_0 and underestimates for large η_0. Clearly, the model can agree with simulations if η_0 happens to be "just right." For example, in Ref. [8] we presented shock tube experiments simulated with CALE for a He-air $M_s = 1.2$ system with $\eta_0^{bubble} = 0.35$ and 0.70 cm. The bubbles agreed, within $\sim 10\%$, with the model [8]. The spikes, on the other hand, deviate substantially: The model overestimates (underestimates) the spike for 0.35 cm (0.70 cm), as shown in Fig. 3. A calculation (not shown) with 0.525 cm showed reasonable agreement for both bubble *and* spike, but it is purely accidental.

The failure for spikes becomes perhaps obvious for this model if we study Eq. (4): Performing Goncharov's transformation and taking the $\eta \rightarrow -\infty$ limit we see that η_2^{spike} asymptotes to $ck/4(1+c)$, which is the *same* as for the bubble (except the sign, of course). Now, it is well known that

spikes are "sharper," i.e., have larger $|\eta_2|$ than bubbles, especially at large A. Equation (4) is independent of A and clearly predicts the wrong curvature for Goncharov's spikes.

What about $A=1$? We find that the models differ even in this case. In other words, while Eq. (6) reduces to Eq. (3) for $A=1$ assuring that the model for bubbles is continuous in A, it does *not* reduce to Eq. (3) after performing Goncharov's transformation and then setting $A=1$. Again, we find this approach to be deficient. Zhang's approach appears to be quite successful. For RT, it is a curious fact, no doubt of little value, that $\ddot{\eta}^{spike} \rightarrow -g$ for both Zhang's and Goncharov's approaches, though they differ at early times. For RM, η^{spike} in one model can be larger or smaller than the other model, again depending on η_0. We shall not consider Goncharov's model for spikes any further.

Zhang's model [6] for 2D spikes was the same as for bubbles except $\eta_0^{spike} < 0$. We generalize this approach to 3D: Eqs. (3) and (4), derived [5] originally for bubbles, may be used for bubbles ($\eta_0 > 0$) or spikes ($\eta_0 < 0$) in 2D ($c=2$) or 3D ($c=1$). Thus we find that $\eta_2^{spike} \rightarrow +\infty$ (exponentially) as $\eta^{spike} \rightarrow -\infty$ and, as mentioned above, $\ddot{\eta}^{spike} \rightarrow -g$ for RT. These asymptotes are independent of c and hence agree with Zhang's findings. For RM we find

$$\left(\frac{\dot{\eta}}{\dot{\eta}_0}\right)^2_{\eta \rightarrow -\infty} = \frac{\xi_0 + c/4}{\xi_0 + c/4(1+c)} = \frac{1 - \eta_0 k}{1/(1+c) - \eta_0 k} \quad (11)$$

for the asymptotic spike velocity. All amplitudes in the above equation refer to spikes, i.e., $\eta_0 < 0$ and $\xi_0 \equiv \eta_2(0)/k = -c\eta_0 k/4$.

In [5] we pointed out that asymptotic bubble velocities in 3D are always larger than 2D. The opposite is true for spike velocities: From Eq. (11) $\dot{\eta}_\infty(3D)/\dot{\eta}_\infty(2D)$ varies between $\sqrt{2/3}$ and 1 as ξ_0 varies from 0 to ∞. To calculate the time evolution of RM spikes in 2D and 3D use the analytic solutions given in [5] for $\eta_0 < \eta^*$, i.e., Eq. (11) and Eq. (14), respectively, of Ref. [5] with a negative η_0.
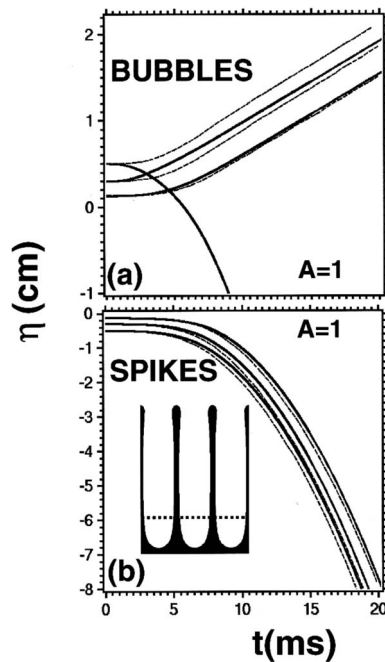
FIG. 4. RT bubbles and spikes in a downward-moving LEM "experiment" with perturbations of $\eta_0=0.13$, 0.3, or 0.5 cm on the surface of the water-NaI solution. Thick continuous lines are from the model (see text) and thin dashed lines from CALE simulations. The light fluid is air. The inset from the 0.5 cm run shows the fluid at 18 ms. Bubbles and spikes are measured relative to the nominal (unperturbed) surface indicated by the dashed line.

Interestingly, Fig. 1 shows an upper limit on η_0 but no lower limit, suggesting that for large $|\eta_0 k|$ the model may be used for spikes but *not* for bubbles. This is the exact opposite of what Layzer advocated—his model was good for bubbles but not for spikes. Of course he had set $\eta_0=0$ so this situation could not arise. To check, we replaced the low density fluid (hexane) by air in our simulations of the LEM experiment [8] so $A \sim 1$, and the results are shown in Fig. 4 for $\eta_0^{bubble}=0.13$, 0.3, and 0.5 cm, using $\eta_0^{spike}=-\eta_0^{bubble}$ following Zhang. The first two simulations exhibit fair agreement with the model for both bubble and spike. As for $\eta_0 =0.5$ cm, it is above $(\eta_0)_{max}$ ($=1/k=0.39$ cm) and indeed the bubble solution fails while the spike solution appears reasonably well reproduced.

Conclusions. For any A one may use the Layzer model for bubbles as long as $\eta_0 < (\eta_0)_{max}$ given in Eq. (8). In addition, the acceleration history must not admit sign-changing solutions to the linear equation, Eq. (1). For any A Goncharov's model [7] for spikes more often than not gives erroneous results, although they are not as patently wrong as when bubbles violate the $(\eta_0)_{max}$ condition. Zhang's model for spikes [6], valid for $A=1$ only, appears to be quite reasonable (we performed 3D simulations and obtained similar results), leading to the unexpected situation that for $|\eta_0| \geq (\eta_0)_{max}$ we have a model for spikes but not for bubbles.

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